How to lift the burden of intertemporal optimization from consumers to the policymaker*

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1 The Ramsey consumers

Ignoring depreciation and population growth, the optimal allocation satisfies the usual Keynes-Ramsey rule and resource constraint (together with the appropriate transversality condition):

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left(f'(k) - \delta - \rho\right), \tag{1}$$

$$\dot{k} = f(k) - c - \delta k, \tag{2}$$

where c is consumption, $\theta^{-1} > 0$ is the rate of intertemporal substitution, $k > 0$ is physical capital, $\delta > 0$ is the depreciation rate of capital, f is an increasing, strictly concave function satisfying the Inada conditions, and $\rho > 0$ is the subjective rate of interest.

2 A simple consumption rule

A simple consumption rule makes life easy for consumers—just as life is easy for policymakers under a simple policy rule. Assume therefore that consumption is given as

$$c = \alpha\left[f(k) - T\right], \tag{3}$$

where T is a lump-sum tax. Hence, it instructs consumers to consume a constant fraction α of their disposable income.[1] Period. No more intertemporal optimization to worry about. The benevolent policymaker, on the other hand, should spend some intellectual resources:

*Addendum to 2011 blog post. Not to be taken scientifically serious.

[1] Underlying all this, i.e., the derivation of (1) and (2), we have the conventional structure of a consumer budget constraint: $\dot{b} = rb + c + i + T - f(k)$, a capital accumulation definition: $\dot{k} = i - \delta k$, and a public budget constraint: $\dot{b} = rb - T$, where b is public debt, i is investment in physical capital, and r is the real interest rate. Optimal intertemporal choice of consumption when utility is $\int_{t=0}^{\infty} e^{-\rho t} \frac{1}{1-\theta} c^{1-\theta} dt$, combined with the relevant equilibrium conditions, then yields these two conditions.

It should find a value of α together with a dynamic adjustment of T, which are able of replicating the optimal allocation, given that the consumers are relieved of performing dynamic optimization, i.e., given that they adhere to (3).

If we think all consumers could do it, then surely the policymaker can. But we save quite some brain power in the population, at the potential expense of some policymaker brain power. Taking the model literally, this should be a cost-benefit problem with an easy solution. So let us put the policymaker to work.

3 Optimal policy under a simple consumption rule

The policymaker should do as follows. As (2) holds by definition, it is a matter of inducing an optimal consumption path, i.e., induce (1). Now, time-differentiating (3) yields

$$\dot{c} = \alpha \left[f'(k) \dot{k} - \dot{T} \right].$$

Use (2) to eliminate $\dot{k}$:

$$\begin{aligned} \dot{c} &= \alpha \left[f'(k) (f(k) - c - \delta k) - \dot{T} \right] \\ &= \alpha \left[f'(k) (f(k) - \alpha [f(k) - T] - \delta k) - \dot{T} \right] \\ &= \alpha \left[f'(k) ((1-\alpha) f(k) + \alpha T - \delta k) - \dot{T} \right], \end{aligned}$$

where the second equality follows by use of (3). Divide through by (3) to get consumption growth as

$$\frac{\dot{c}}{c} = \frac{f'(k)((1-\alpha) f(k) + \alpha T - \delta k) - \dot{T}}{f(k) - T}.$$

Hence, policy is optimal if

$$\frac{1}{\theta}(f'(k) - \delta - \rho) = \frac{f'(k)((1-\alpha) f(k) + \alpha T - \delta k) - \dot{T}}{f(k) - T}. \tag{4}$$

This can easily be achieved. Alternatively, take α as given, and maximize

$$U = \int_{t=0}^{\infty} e^{-\rho t} \frac{1}{1-\theta} \alpha^{1-\theta} (f(k) - T)^{1-\theta} \mathrm{d}t$$

subject to $\dot{k} = f(k) - \alpha(f(k) - T) - \delta k$. The Hamiltonian is

$$\mathcal{H} = e^{-\rho t} \frac{1}{1-\theta} \alpha^{1-\theta} (f(k) - T)^{1-\theta} + \lambda [f(k) - \alpha(f(k) - T) - \delta k]$$

The first-order conditions are

$$\begin{aligned} e^{-\rho t} \alpha^{1-\theta} (f(k) - T)^{-\theta} &= \lambda \alpha, \\ e^{-\rho t} \alpha^{1-\theta} (f(k) - T)^{-\theta} f'(k) + \lambda [(1-\alpha) f'(k) - \delta] &= -\dot{\lambda}, \end{aligned}$$

and thus

$$\begin{aligned} e^{-\rho t} \alpha^{1-\theta} (f(k) - T)^{-\theta} &= \lambda \alpha, \\ \alpha f'(k) + (1-\alpha) f'(k) - \delta &= -\frac{\dot{\lambda}}{\lambda}, \end{aligned}$$

leading to

$$
\begin{aligned}
e^{-\rho t}\alpha^{1-\theta}\left(f\left(k\right)-T\right)^{-\theta} &= \lambda\alpha, \\
f'\left(k\right)-\delta &= -\frac{\dot{\lambda}}{\lambda}.
\end{aligned}
$$

Time-differentiation of the first equation leads to

$$
-\rho e^{-\rho t}\alpha^{1-\theta}\left(f\left(k\right)-T\right)^{-\theta}-\theta e^{-\rho t}\alpha^{1-\theta}\left(f\left(k\right)-T\right)^{-\theta-1}\left(f'\left(k\right)\dot{k}-\dot{T}\right)=\dot{\lambda}\alpha,
$$

and thus

$$
-\rho-\theta\frac{f'\left(k\right)\dot{k}-\dot{T}}{f\left(k\right)-T}=\frac{\dot{\lambda}}{\lambda},
$$

leading finally to

$$
\frac{f'\left(k\right)\dot{k}-\dot{T}}{f\left(k\right)-T}=\frac{1}{\theta}\left[f'\left(k\right)-\delta-\rho\right],
$$

which is, of course, the same as (4). This states, not surprisingly, that T should be adjusted such that disposable income grows at the rate by which consumption grows under optimization by private agents.

What determines then α? The associated optimal $c\left(0\right)$. Given this, we find α from

$$
c\left(0\right)=\alpha\left[f\left(k\left(0\right)\right)-T\left(0\right)\right].
$$

I.e., α is chosen at will, and $T\left(0\right)$ then jumps so as to achieve $c\left(0\right)$. Thereafter, given α, T and k must obey the system of differential equations

$$
\begin{aligned}
\frac{f'\left(k\right)\dot{k}-\dot{T}}{f\left(k\right)-T} &= \frac{1}{\theta}\left[f'\left(k\right)-\delta-\rho\right] \\
\dot{k} &= f\left(k\right)-\alpha\left(f\left(k\right)-T\right)-\delta k
\end{aligned}
$$

(This is saddle-path stable, as one, of course, can define $f\left(k\right)-T\equiv x/\alpha$, such that the system becomes

$$
\begin{aligned}
\frac{\dot{x}}{x} &= \frac{1}{\theta}\left[f'\left(k\right)-\delta-\rho\right], \\
\dot{k} &= f\left(k\right)-x-\delta k,
\end{aligned}
$$

which are the usual Ramsey dynamics with x substituted for c.)